

# SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS IN TUBES

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**ABSTRACT.** Given a smooth compact  $k$ -dimensional manifold  $\Lambda$  embedded in  $\mathbb{R}^m$ , with  $m \geq 2$  and  $1 \leq k \leq m-1$ , and given  $\epsilon > 0$ , we define  $B_\epsilon(\Lambda)$  to be the geodesic tubular neighborhood of radius  $\epsilon$  about  $\Lambda$ . In this paper, we construct positive solutions of the semilinear elliptic equation

$$\begin{cases} \Delta u + u^p &= 0 & \text{in } B_\epsilon(\Lambda) \\ u &= 0 & \text{on } \partial B_\epsilon(\Lambda), \end{cases}$$

when the parameter  $\epsilon$  is chosen small enough. In this equation, the exponent  $p$  satisfies either  $p > 1$  when  $n := m - k \leq 2$  or  $p \in (1, \frac{n+2}{n-2})$  when  $n > 2$ . In particular  $p$  can be critical or supercritical in dimension  $m \geq 3$ . As  $\epsilon$  tends to 0, the solutions we construct have Morse index tending to infinity. Moreover, using a Pohozaev type argument, we prove that our result is sharp in the sense that there are no positive solutions for  $p > \frac{n+2}{n-2}$ ,  $n \geq 3$ , if  $\epsilon$  is sufficiently small.

## 1. INTRODUCTION

Assume that we are given  $\Lambda$ , a smooth compact  $k$ -dimensional submanifold which is embedded in  $\mathbb{R}^m$ , where  $m \geq 2$  and  $k \in \{1, \dots, m-1\}$ . For all  $\epsilon > 0$ , we define the plain tubular neighborhood of radius  $\epsilon$ , centered about  $\Lambda$  by

$$(1.1) \quad B_\epsilon(\Lambda) := \{x \in \mathbb{R}^m : \text{dist}(x, \Lambda) < \epsilon\},$$

where  $x \mapsto \text{dist}(x, Y)$  denotes the Euclidean distance in  $\mathbb{R}^m$  from  $x$  to  $Y$ . Let us observe that, for all  $\epsilon$  small enough, the boundary of  $B_\epsilon(\Lambda)$ , which is defined by

$$T_\epsilon(\Lambda) := \{x \in \mathbb{R}^m : \text{dist}(x, \Lambda) = \epsilon\},$$

is a smooth embedded hypersurface in  $\mathbb{R}^m$ .

The aim of the paper is to show, for  $\epsilon$  small, the existence of a new family of positive solutions of the semilinear elliptic problem

$$(1.2) \quad \begin{cases} \Delta u + u^p &= 0 & \text{in } B_\epsilon(\Lambda) \\ u &= 0 & \text{on } T_\epsilon(\Lambda). \end{cases}$$

Note that, in this paper  $\Delta$  will always represent the Laplace operator in  $\mathbb{R}^n$ .

To state precisely our result we need some preliminaries. Let

$$n := m - k \geq 1,$$

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denote the codimension of  $\Lambda$  in  $\mathbb{R}^m$ . It is well known that, if  $B_1^n$  denotes the  $n$ -dimensional unit ball in  $\mathbb{R}^n$ , there exists a positive solution  $U$  of

$$(1.3) \quad \begin{cases} \Delta U + U^p = 0 & \text{in } B_1^n \\ U = 0 & \text{on } \partial B_1^n, \end{cases}$$

provided the exponent  $p$  is chosen to satisfy  $1 < p < +\infty$  when  $n \leq 2$  or  $p \in (1, \frac{n+2}{n-2})$  when  $n \geq 3$ . Thanks to Gidas-Ni-Nirenberg's Theorem [8], the function  $U$  is known to be radially symmetric. Moreover, this solution is known to be unique, nondegenerate (we refer to Theorem 4.1 and Theorem 4.2 in [6] for a proof of this fact) and to have Morse index equal to 1.

For all  $\epsilon > 0$ , we define

$$(1.4) \quad \bar{u}_\epsilon(x) := \epsilon^{-\frac{2}{p-1}} U\left(\frac{\text{dist}(x, \Lambda)}{\epsilon}\right),$$

for all  $x \in B_\epsilon(\Lambda)$ . This function is obtained by translating a rescaled copy of  $U$  along the manifold  $\Lambda$ . With these notations at hand, we have the following :

**Theorem 1.1.** *Assume that*

$$p \in (1, +\infty), \quad \text{if } n \leq 2, \quad \text{or } p \in \left(1, \frac{n+2}{n-2}\right), \quad \text{if } n \geq 3,$$

where  $n := m - k$ . Then, there exists  $\bar{\epsilon} > 0$  and  $\mathcal{S} \subset (0, \bar{\epsilon})$  such that :

(1) For all  $\epsilon \in \mathcal{S}$ , there exists a positive solution  $u_\epsilon$  of (1.2) satisfying

$$(1.5) \quad \lim_{\epsilon \rightarrow 0, \epsilon \in \mathcal{S}} \left\| \frac{u_\epsilon}{\bar{u}_\epsilon} \right\|_{L^\infty(B_\epsilon(\Lambda))} = 1.$$

(2) For all  $\alpha \geq 1$ ,

$$(1.6) \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^\alpha} (\epsilon - \text{meas } \mathcal{S}_\epsilon) = 0,$$

where  $\mathcal{S}_\epsilon := \mathcal{S} \cap (0, \epsilon)$ .

(3) As  $\epsilon \in \mathcal{S}$  tends to 0, the Morse index of  $u_\epsilon$  tends to infinity.

Let us briefly comment on our result and in particular on the structure of the set  $\mathcal{S}$  in which the parameter  $\epsilon$  can be chosen. As will be apparent in the proof, our construction does not hold for all values of the parameter  $\epsilon$  close to 0. There is a resonance phenomenon which prevents the construction to hold for any small value of  $\epsilon$  and which forces  $\epsilon$  to be taken away from a set of small density close to 0. This is precisely the meaning of (1.6). Such a phenomenon is not new and, in the context of semilinear partial differential equations, it was originally found by A. Malchiodi and M. Montenegro in [13]. Since this seminal paper, this phenomenon has also been found in other instances, for example in the study of other semilinear partial differential equations [11, 14] or in the study of constant mean curvature surfaces [12, 15].

The Morse index of  $u_\epsilon$  is defined to be equal to the dimension of the subspace of  $H_0^1(B_\epsilon(\Lambda))$  over which the quadratic form

$$v \mapsto \int_{B_\epsilon(\Lambda)} (|\nabla v|^2 - p u_\epsilon^{p-1} v^2) dx,$$

is definite negative. The fact that we are not able to construct the solutions for all values of  $\epsilon$  close enough to 0 is also reflected in another important feature of our solutions, namely that their Morse index tends to infinity as  $\epsilon$  tends to 0.

In the same way, recall that when  $p = \frac{m+2}{m-2}$  and  $k = m - 1$  it has been proved in [4] that the energy and the Morse index of all positive solutions tend to infinity as  $\epsilon$  tends to 0.

The shape of the solution we construct is also worth mentioning, in fact, the solution  $u_\epsilon$  is close to the function  $\bar{u}_\epsilon$  which has been defined in (1.4) and hence it does not concentrate at points as  $\epsilon \rightarrow 0$ . Also, by (1.4), we have

$$\|\bar{u}_\epsilon\|_{L^\infty(B_\epsilon(\Lambda))} = \mathcal{O}(\epsilon^{-\frac{2}{p-1}}),$$

as  $\epsilon$  tends to 0.

In the particular case where the exponent  $p$  is the critical Sobolev exponent i.e. when  $p = \frac{m+2}{m-2}$ , a well known theorem by A. Bahri and J.M. Coron [2] yields the existence of positive solutions of (1.2), provided that the topology of the domain is not trivial. In this case, our result can be seen as a direct construction of a positive solution, via a technique which also gives the shape of the solution, that cannot be deduced from the proof in [2].

The solutions we construct are also new in the subcritical case ( $p < \frac{m+2}{m-2}$ ) since they are qualitatively different from the so called multibump solutions which were found in [1, 7] and which do not satisfy (1.5).

Let us observe that the result of Theorem 1.1 holds for supercritical exponents, namely exponents which are larger than the critical Sobolev exponent  $\frac{m+2}{m-2}$  in dimension  $m$ . In the particular case where the codimension  $n$  of the manifold  $\Lambda$  is equal to 1 or 2, the exponent  $p$  can be taken to be arbitrarily large. To our knowledge this is the first existence result for solutions of (1.2) defined in tubular neighborhoods of general  $k$ -dimensional manifolds. A previous result has been recently obtained for  $(m - 1)$ -dimensional manifolds in [3] and it was indeed a source of inspiration for the present paper.

As it will become clear in the proof, the smoothness of the submanifold  $\Lambda$  is a key ingredient of the proof. However, close inspection also shows that this assumption can be relaxed if one is ready to loose some control on the density of the set  $\mathcal{S}$  close to 0. Indeed, we have the :

**Proposition 1.1.** *Under the assumptions of Theorem 1.1, if in point (2) of Theorem 1.1 we fix  $\alpha \geq 1$ , then there exists  $l \in \mathbb{N}$  only depending on  $n$  and  $\alpha$ , such that the conclusion of Theorem 1.1 holds provided  $\Lambda$  is at least a  $C^l$  submanifold. The larger  $\alpha$  is, the larger  $l$  has to be chosen.*

Let us emphasize on the fact that the scheme of the proof is not new and in fact it is inspired from [13] and [12]. However, our framework is simpler and we hope that this will help the interested reader to understand the ideas and techniques in these more involved works.

When  $n \geq 3$ , the existence result of Theorem 1.1 holds under the assumption that

$$p < \frac{n+2}{n-2}.$$

Note that  $\frac{n+2}{n-2}$  is the critical Sobolev exponent in dimension  $n$  and observe that this assumption is used to construct the approximate solution  $\bar{u}_\epsilon$  to (1.2). One might

wonder whether if this condition is only technical. As we will see this is not the case and the existence of positive solutions to (1.2) generally fails if  $p > \frac{n+2}{n-2}$  as we prove by a Pohozaev type argument.

**Theorem 1.2.** *Assume that  $n = m - k \geq 3$  and that*

$$p > \frac{n+2}{n-2}.$$

*Then, there exists  $\bar{\epsilon} = \bar{\epsilon}(p) > 0$ , such that for all  $\epsilon \in (0, \bar{\epsilon})$  there is no bounded positive solution of (1.2) in  $B_\epsilon(\Lambda)$ .*

The proof of Theorem 1.2 relies on a Pohozaev type identity which we derive for solutions of (1.2). This is a standard techniques which has been used in several nonexistence results and it goes back to [17] where the case of star-shaped domains was considered. A similar idea was already used by D. Passaseo in [16], with a more involved construction, leading to nonexistence results for superlinear elliptic problems in topologically nontrivial domains. In our case, the use of suitable coordinates, namely Fermi coordinates (see Section 3), proves to be extremely useful to get Theorem 1.2 in a simple way. We emphasize that as in [16], our domains are not star-shaped and are not topologically trivial.

## 2. OUTLINE OF THE PROOF OF THEOREM 1.1

The proof of Theorem 1.1 consists in showing that there exists a *genuine solution*  $u_\epsilon$  near the *approximate solution*  $\bar{u}_\epsilon$  defined in (1.4) provided the parameter  $\epsilon$  is chosen small enough and away from a set where resonance occurs. The main steps of the proof are the following :

(i) First, we observe that

$$\|\Delta \bar{u}_\epsilon + \bar{u}_\epsilon^p\|_{L^\infty(B_\epsilon(\Lambda))} \leq C\epsilon^{-\frac{p+1}{p-1}}.$$

Then, using a finite step iteration scheme, we improve the approximate solution  $\bar{u}_\epsilon$  into a sequence of approximate solutions  $(u_{\epsilon,i})_{i \in \mathbb{N}}$ , which are as close as we want from a genuine solution of the equation in the sense that

$$\|\Delta u_{\epsilon,i} + u_{\epsilon,i}^p\|_{L^\infty(B_\epsilon(\Lambda))} \leq C\epsilon^{i-\frac{p+1}{p-1}}.$$

Moreover, the sequence  $(u_{\epsilon,i})_{i \in \mathbb{N}}$  is constructed in such a way that one has a good control on the difference  $u_{\epsilon,i} - \bar{u}_\epsilon$ . As already mentioned, the construction of  $u_{\epsilon,i}$  relies on some iteration scheme and we will see that, in order to keep a good control on the sequence of approximate solutions, we need to allow a loss of regularity at each iteration. In particular, the fact that the sequence  $(u_{\epsilon,i})_{i \in \mathbb{N}}$  exists for all  $i \in \mathbb{N}$ , uses the fact that the manifold  $\Lambda$  is smooth in an essential way. If the submanifold  $\Lambda$  has only finite regularity, the sequence can just be constructed for a finite number of indices.

(ii) Next, we study the linearized operator

$$L_{\epsilon,i} := \Delta + pu_{\epsilon,i}^{p-1},$$

about the approximate solution  $u_{\epsilon,i}$  and show that the norm of the inverse of  $L_{\epsilon,i}$  can be controlled as  $\epsilon$  tends to 0, provided  $\epsilon$  is taken away from a countable sequence tending to 0. More precisely, we will see that the Morse index of  $L_{\epsilon,i}$  tends to infinity as  $\epsilon$  tends to 0. In particular, for fixed  $i \in \mathbb{N}$ ,

- the operator  $L_{\epsilon,i}$  is not invertible for a sequence of values  $\epsilon$  tending to 0, and, in order to proceed, we will need to take  $\epsilon$  away from these values.
- (iii) Finally, we look for a genuine solution of 1.2 in the form

$$u_\epsilon = u_{\epsilon,i} + \varphi_\epsilon.$$

At this stage, we show that we can rephrase the problem as a fixed point problem which can easily be solved using the fixed point theorem for contraction mappings.

The outline of the paper is the following. In section 3 we describe the notations we use and state basic results about the structure of the Laplace operator in Fermi coordinates about  $\Lambda$ . In section 4, we construct the sequence of approximate solutions  $(u_{\epsilon,i})_{i \in \mathbb{N}}$  and derive the relevant estimates. In Section 5 and 6, we analyze the spectrum and the uniform invertibility of  $L_{\epsilon,i}$ , the linearized operator about  $u_{\epsilon,i}$ . In section 7, we complete the proof of Theorem 1.1 by reducing the problem to the solvability of a fixed point problem for contraction mappings. In section 8 we prove Theorem 1.2.

### 3. FERMI COORDINATES NEAR $\Lambda$

An important tool in the proof of Theorem 1.1 is the use of appropriate coordinates to parameterize  $B_\epsilon(\Lambda)$ . We identify  $\Lambda$  with the zero section of  $N\Lambda$  the normal bundle of  $\Lambda$  and  $B_\epsilon(\Lambda)$  will be identified with

$$\Omega_\epsilon(\Lambda) := \{(y, z) \in N\Lambda : y \in \Lambda, \quad z \in N_y\Lambda, \quad |z| < \epsilon\},$$

via the mapping

$$\begin{aligned} F_1 : \Omega_\epsilon(\Lambda) &\rightarrow B_\epsilon(\Lambda) \\ (y, z) &\mapsto y + z. \end{aligned}$$

The normal bundle  $N\Lambda$  is endowed with the metric induced by the embedding of  $\Lambda$  in  $\mathbb{R}^m$ , namely

$$\bar{g} = \mathring{g} + g_z,$$

where  $\mathring{g}$  is the induced metric on  $\Lambda$  and  $g_z := dz^2$  the (Euclidean) metric on the normal fibers.

In a neighborhood of a given point  $y \in \Lambda$ , we can define a moving orthonormal frame

$$e^1, \dots, e^n \in N\Lambda,$$

where each  $e^j$  is a smooth section of the normal bundle  $N\Lambda$ . Namely, locally the vectors  $e^1(y), \dots, e^n(y)$  constitute an orthonormal basis of the normal space to  $\Lambda$  at  $y$  and  $y \mapsto e^j(y)$  is a smooth vector field. A moving orthonormal frame might not be globally defined but it is always defined in a neighborhood of a given point in  $\Lambda$ .

We can then define  $\Phi$ , a local parametrization from a neighborhood of  $(y, 0) \in \Lambda \times \mathbb{R}^n$  into a neighborhood of  $y \in \mathbb{R}^m$ , by

$$\Phi(y, z_1, \dots, z_n) := y + \sum_{i=1}^n z_i e^i(y),$$

and  $(y, z_1, \dots, z_n)$  will be referred to as Fermi coordinates. In this parametrization, the Euclidean metric

$$(3.7) \quad g_\circ := dx_1^2 + \dots + dx_m^2,$$

in  $\mathbb{R}^m$ , or more precisely  $\Phi^*g_\circ$ , the pull back of  $g_\circ$  by  $\Phi$ , is close to  $\bar{g}$  the induced metric on  $N\Lambda$ . The next Lemma gives a quantitative version of this statement.

**Lemma 3.1.** *In the above defined coordinates,*

$$(3.8) \quad \Phi^*g_\circ := \bar{g} + \sum_{i=1}^n z_i(\mathring{h}^i + 2\mathring{\ell}^i) + \sum_{i,j=1}^n z_i z_j \mathring{k}^{ij},$$

where the tensors  $\mathring{h}^i$ ,  $\mathring{\ell}^i$  and  $\mathring{k}^{ij}$  acting on  $T\Lambda$ , have coefficients which are smooth functions on  $\Lambda$ .

*Proof.* We denote by

$$(t_1, \dots, t_k) \mapsto Y(t_1, \dots, t_k),$$

a parametrization of  $\Lambda$  close to a given point  $y_\bullet$  and, without loss of generality, we assume that  $Y(0) = y_\bullet$ .

To keep notations short, we agree that  $e^j \circ Y$  is also denoted by  $e^j$ . Hence

$$X(t_1, \dots, t_k, z_1, \dots, z_n) := Y(t_1, \dots, t_k) + \sum_{i=1}^n z_i e^i(t_1, \dots, t_k),$$

is a parametrization on  $\mathbb{R}^m$  close to  $y_\bullet$ . To compute the coefficients of the Euclidean metric in these coordinates, it is enough to compute

$$\partial_{t_a} X \cdot \partial_{t_b} X, \quad \partial_{t_a} X \cdot \partial_{z_j} X \quad \text{and} \quad \partial_{z_i} X \cdot \partial_{z_j} X.$$

Observe that  $\partial_{t_a} Y$  is a tangent vector to  $\Lambda$  while  $e^j$  is a normal vector to  $\Lambda$  and hence  $\partial_{t_a} Y \cdot e^j \equiv 0$ . Using this, it is easy to check that

$$\partial_{t_a} X \cdot \partial_{t_b} X = \partial_{t_a} Y \cdot \partial_{t_b} Y + \sum_{i=1}^n z_i (\partial_{t_a} Y \cdot \partial_{t_b} e^i + \partial_{t_b} Y \cdot \partial_{t_a} e^i) + \sum_{i,j=1}^n z_i z_j \partial_{t_a} e^i \cdot \partial_{t_b} e^j,$$

$$\partial_{t_a} X \cdot \partial_{z_j} X = \sum_{i=1}^n z_i \partial_{t_a} e^i \cdot e^j \quad \text{and} \quad \partial_{z_i} X \cdot \partial_{z_j} X = \sum_{i,j=1}^n e^i \cdot e^j.$$

We set

$$\mathring{h}^i := \sum_{a,b=1}^k (\partial_{t_a} Y \cdot \partial_{t_b} e^i + \partial_{t_b} Y \cdot \partial_{t_a} e^i) dt_a dt_b,$$

$$\mathring{k}^{ij} := \sum_{a,b=1}^k \partial_{t_a} e^i \cdot \partial_{t_b} e^j dt_a dt_b \quad \text{and} \quad \mathring{\ell}^i := \sum_{a=1}^k \sum_{j=1}^n \partial_{t_a} e^i \cdot e^j dt_a dz_j.$$

Observe that these are smooth functions defined on  $\Lambda$ .

With these notations at hand, we can write

$$\begin{aligned} g_\circ &= \sum_{a,b=1}^k \left( \mathring{g}_{ab} + \sum_{i=1}^n z_i \mathring{h}_{ab}^i + \sum_{i,j=1}^n z_i z_j \mathring{k}_{ab}^{ij} \right) dt_a dt_b \\ &+ 2 \sum_{a=1}^k \sum_{j=1}^n \left( \sum_{i=1}^n z_i \mathring{\ell}_{aj}^i \right) dt_a dz_j + \sum_{i=1}^n dz_i^2, \end{aligned}$$

so the proof of (3.8) is completed.  $\square$

Recall that, if on a given manifold  $M$  the metric tensor is given in local coordinates by

$$g = \sum_{i,j=1}^m g_{ij} dx_i dx_j,$$

then the Laplace-Beltrami operator is given by

$$\Delta_g := \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^m \partial_{x_i} \left( \sqrt{\det g} g^{ij} \partial_{x_j} \cdot \right),$$

where  $g^{ij}$  are the coefficients of the inverse of the matrix  $(g_{ij})_{ij}$ .

Using this formula, together with the expansion in Lemma 3.1, we get the :

**Lemma 3.2.** *In a tubular neighborhood of  $\Lambda$ , the Euclidean Laplacian  $\Delta$  can be decomposed as*

$$\Delta := \sum_{i=1}^m \partial_{x_i}^2 = \Delta_{\bar{g}} + D,$$

where  $\Delta_{\bar{g}} = \Delta_{\bar{g}} + \Delta_{g_z}$  denotes the Laplace-Beltrami operator on  $N\Lambda$  for the metric  $\bar{g} = \bar{g} + g_z$ , and  $D$  is a second order differential operator which, in Fermi coordinates, can be expanded as

$$D = \sum_{i=1}^n z_i D_i^{(2)} + D^{(1)},$$

where  $D_i^{(2)}$  (respectively  $D^{(1)}$ ) are second order (respectively first order) partial differential operators whose coefficients are smooth and bounded in some fixed tubular neighborhood of  $\Lambda$ .

*Proof.* The proof follows from the result of Lemma 3.1 and the expression of the Laplacian in local coordinates.  $\square$

We can define, in a fixed tubular neighborhood of  $\Lambda$ , the function  $a$  by

$$(3.9) \quad d\text{vol}_{g_o} = a d\text{vol}_{\bar{g}}.$$

Observe that  $a$  is smooth and  $a \equiv 1$  along  $\Lambda$ . Moreover, it follows from Lemma 3.1, that there exists a constant  $C > 0$  such that

$$|a - 1| \leq C |z|,$$

in a tubular neighborhood of  $\Lambda$ .

#### 4. CONSTRUCTION OF A SEQUENCE OF APPROXIMATE SOLUTIONS

We assume that  $p \in (1, +\infty)$  if  $n = 1, 2$  or  $p \in (1, \frac{n+2}{n-2})$  if  $n \geq 3$ . As in the introduction, we let  $U$  to be the unique positive radial solution of (1.3) and we set

$$(4.10) \quad L := -(\Delta + pU^{p-1}),$$

to be the linearized operator about  $U$ . The spectrum of  $L$  will be denoted by

$$(4.11) \quad \mu_0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$$

and the corresponding eigenfunctions, which will be denoted by  $\phi_j$ , are normalized to have norm 1 in  $L^2(B_1^n)$ . It is known (see Theorem 4.1 and Theorem 4.2 in [6]) that

$$\mu_0 < 0 < \mu_1.$$

We use the Fermi coordinates introduced in the previous section. According to (1.4), the function  $\bar{u}_\epsilon$  only depends on  $|z|$  which is nothing but the distance function from a point to  $\Lambda$ . We can write

$$\Delta \bar{u}_\epsilon + \bar{u}_\epsilon^p = (\Delta_{g_z} \bar{u}_\epsilon + \bar{u}_\epsilon^p) + (\Delta_{\dot{g}} + D) \bar{u}_\epsilon.$$

Since  $\Delta_{\dot{g}} \bar{u}_\epsilon = 0$  and since  $\Delta_{g_z} \bar{u}_\epsilon + \bar{u}_\epsilon^p = 0$ , we conclude that

$$(4.12) \quad \Delta \bar{u}_\epsilon + \bar{u}_\epsilon^p = D \bar{u}_\epsilon.$$

As mentioned before, the idea is first to implement an iteration scheme to perturb  $\bar{u}_\epsilon$  into a sequence of approximate solutions which are closer to being a genuine solution of our problem. To do so, we write  $u = \bar{u}_\epsilon + v$  and, making use of the result of Lemma 3.2 and (4.12), we rewrite the equation in (1.2) as

$$(4.13) \quad -(\Delta_{g_z} + p \bar{u}_\epsilon^{p-1})v = E_\epsilon + K_\epsilon(v) + (\Delta_{\dot{g}} + D)v,$$

where by definition

$$E_\epsilon := \Delta \bar{u}_\epsilon + \bar{u}_\epsilon^p,$$

and

$$K_\epsilon(v) := |\bar{u}_\epsilon + v|^p - \bar{u}_\epsilon^p - p \bar{u}_\epsilon^{p-1} v.$$

The iteration scheme we use is the following : we set  $v_{\epsilon,0} \equiv 0$  and, for all  $i \geq 0$ , we define inductively  $v_{\epsilon,i+1}$  to be the solution of

$$(4.14) \quad \begin{cases} -(\Delta_{g_z} + p \bar{u}_\epsilon^{p-1})v_{\epsilon,i+1} &= E_\epsilon + K_\epsilon(v_{\epsilon,i}) + (\Delta_{\dot{g}} + D)v_{\epsilon,i} & \text{in } B_\epsilon(\Lambda) \\ v_{\epsilon,i+1} &= 0 & \text{on } T_\epsilon(\Lambda). \end{cases}$$

Observe that the functions are defined in  $B_\epsilon(\Lambda)$  but the operator on the left hand side only depends on the variable normal to  $\Lambda$ , namely  $\partial_{z_j}$ . So, when we solve this equation, we only solve the equation in  $B_\epsilon^n$ , the ball of radius  $\epsilon$  in  $\mathbb{R}^n$  centered at the origin, and we consider the variable on  $\Lambda$  as parameters. At each iteration, we loose two degrees of regularity in the variables belonging to  $\Lambda$  but this is not a problem if we assume that  $\Lambda$  is differentiable enough, in fact this is where we need  $\Lambda$  to be smooth if we want the sequence to be defined for all  $i$  and  $\Lambda$  should be regular enough if we just need a finite number of iteration. For sake of simplicity, we state and prove all results when  $\Lambda$  is a smooth submanifold of  $\mathbb{R}^m$ , leaving the statement for the case where  $\Lambda$  has finite smoothness to the reader.

To invert the left hand side of (4.13), we simply use a scaling argument and the fact that, according to the result in [6], 0 is not in the spectrum of the operator  $L$  and hence this operator is invertible. In particular, if one wants to solve

$$\begin{cases} -(\Delta + p \bar{u}_\epsilon^{p-1})v &= f & \text{in } B_\epsilon^n \\ v &= 0 & \text{on } \partial B_\epsilon^n, \end{cases}$$

one just considers  $\tilde{v}(x) := v(\epsilon x)$  and  $\tilde{f}(x) := f(\epsilon x)$  which solve

$$\begin{cases} -(\Delta + p U^{p-1})\tilde{v} &= \epsilon^2 \tilde{f} & \text{in } B_1^n \\ \tilde{v} &= 0 & \text{on } \partial B_1^n. \end{cases}$$

Standard elliptic estimates for  $\tilde{v}$  are available and the corresponding scaled estimates for the function  $v$  follow at once. Observe the gain of two powers of  $\epsilon$  due to the presence of  $\epsilon^2$  on the right hand side of the last equation.



We define

$$(4.15) \quad u_{\epsilon,i} := \bar{u}_\epsilon + v_{\epsilon,i}.$$

We claim the following :

**Proposition 4.1.** *There exist constants  $C > 0$  and  $\epsilon_0 > 0$  such that, for all  $\epsilon \in (0, \epsilon_0)$  and for all  $i \in \mathbb{N}$*

$$(4.16) \quad \|\Delta u_{\epsilon,i} + u_{\epsilon,i}^p\|_{L^\infty(B_\epsilon(\Lambda))} \leq C\epsilon^{i-\frac{p+1}{p-1}},$$

and

$$(4.17) \quad \left\| \frac{v_{\epsilon,i}}{\bar{u}_\epsilon} \right\|_{L^\infty(B_\epsilon(\Lambda))} + \epsilon \left\| \frac{\partial_\epsilon v_{\epsilon,i}}{\bar{u}_\epsilon} \right\|_{L^\infty(B_\epsilon(\Lambda))} \leq C\epsilon.$$

*Proof.* The proof is decomposed into a few steps each of which takes advantage of a particular property of the problem we are studying. To begin, we need to introduce the norms

$$\|u\|_{C_\epsilon^{0,\alpha}(B_\epsilon(\Lambda))} := \|u\|_{L^\infty(B_\epsilon(\Lambda))} + \sup_{(y,z),(y,z') \in \Omega_\epsilon(\Lambda)} \epsilon^\alpha \frac{|u \circ F_1(y,z) - u \circ F_1(y,z')|}{|z - z'|^\alpha},$$

and

$$\begin{aligned} \|u\|_{C_\epsilon^{2,\alpha}(B_\epsilon(\Lambda))} &:= \|u\|_{L^\infty(B_\epsilon(\Lambda))} + \epsilon \|\nabla_{g_z} u\|_{L^\infty(B_\epsilon(\Lambda))} + \epsilon^2 \|\nabla_{g_z}^2 u\|_{L^\infty(B_\epsilon(\Lambda))} \\ &+ \sup_{(y,z),(y,z') \in \Omega_\epsilon(\Lambda)} \epsilon^{2+\alpha} \frac{|\nabla_{g_z}^2 u \circ F_1(y,z) - \nabla_{g_z}^2 u \circ F_1(y,z')|}{|z - z'|^\alpha}. \end{aligned}$$

**Step 1.** According to (4.12), we have in local coordinates

$$\Delta \bar{u}_\epsilon + \bar{u}_\epsilon^p = \sum_{i=1}^n z_i D_i^{(2)} \bar{u}_\epsilon + D^{(1)} \bar{u}_\epsilon.$$

As remarked earlier, the function  $\bar{u}_\epsilon$  only depends on  $|z|$  and hence we have

$$\|\bar{u}_\epsilon\|_{C_\epsilon^{2,\alpha}(B_\epsilon(\Lambda))} \leq C\epsilon^{-\frac{2}{p-1}}.$$

This follows from the fact that  $U \in \mathcal{C}^{2,\alpha}(B_1^n)$ .

Taking advantage of the fact that the coefficients in  $D_i^{(2)}$  and  $D^{(1)}$  are smooth functions whose partial derivatives are bounded independently of  $\epsilon$ , we conclude that for all  $\ell \in \mathbb{N}$ , we have

$$\epsilon \|\nabla_{\tilde{g}}^\ell D_i^{(2)} \bar{u}_\epsilon\|_{C_\epsilon^{0,\alpha}(B_\epsilon(\Lambda))} + \|\nabla_{\tilde{g}}^\ell D^{(1)} \bar{u}_\epsilon\|_{C_\epsilon^{0,\alpha}(B_\epsilon(\Lambda))} \leq C_\ell \epsilon^{-\frac{p+1}{p-1}}.$$

for some constant  $C_\ell > 0$  which does not depend on  $\epsilon \in (0, 1)$ . This implies that, for all  $\ell \in \mathbb{N}$ , there exists  $C_\ell > 0$  such that, for all  $\epsilon \in (0, 1)$

$$\|\nabla_{\tilde{g}}^\ell (\Delta \bar{u}_\epsilon + \bar{u}_\epsilon^p)\|_{C_\epsilon^{0,\alpha}(B_\epsilon(\Lambda))} \leq C_\ell \epsilon^{-\frac{p+1}{p-1}}.$$

Taking  $\ell = 0$ , this already proves (4.16) when  $i = 0$ .

**Step 2.** To prove the first half of (4.17) when  $i = 1$ , we use the fact that

$$-(\Delta_{g_z} + p\bar{u}_\epsilon^{p-1})v_{\epsilon,1} = \Delta \bar{u}_\epsilon + \bar{u}_\epsilon^p.$$

Using the inverse of the operator  $-(\Delta_{g_z} + p\bar{u}_\epsilon^{p-1})$  and considering the variables on  $\Lambda$  as parameters we get from standard elliptic estimates that there exists  $C_\ell > 0$  such that, for all  $\epsilon \in (0, 1)$

$$\|v_{\epsilon,1}\|_{C_\epsilon^{2,\alpha}(B_\epsilon(\Lambda))} \leq C\epsilon^{1-\frac{2}{p-1}}.$$

We have obtained the first half of (4.17) when  $i = 1$ .

**Step 3.** We now derive some estimates for the partial derivatives of  $v_{\epsilon,1}$  in the direction parallel to  $\Lambda$ . More precisely, we choose local coordinates  $t_1, \dots, t_k$  on  $\Lambda$  and a cutoff function  $\chi$  with compact support where these coordinates are well defined. Observe that the operator

$$\Gamma : f \mapsto \chi \partial_{t_{i_1}} \dots \partial_{t_{i_\ell}} f,$$

commutes with  $-(\Delta_{g_z} + p\bar{u}_\epsilon^{p-1})$  and hence

$$-(\Delta_{g_z} + p\bar{u}_\epsilon^{p-1})\Gamma v_{\epsilon,1} = \Gamma(\Delta \bar{u}_\epsilon + \bar{u}_\epsilon^p).$$

Moreover, since  $v_{\epsilon,1}$  vanishes on  $T_\epsilon(\Lambda)$ , so does  $\Gamma v_{\epsilon,1}$ . Since we have already estimated the right hand side of this equation, we can use the inverse of the operator  $-(\Delta_{g_z} + p\bar{u}_\epsilon^{p-1})$  and we get, for all  $\ell \in \mathbb{N}$ ,

$$(4.18) \quad \|\nabla_{\dot{g}}^\ell v_{\epsilon,1}\|_{C_\epsilon^{2,\alpha}} \leq C_\ell \epsilon^{1-\frac{2}{p-1}},$$

where, as usual,  $C_\ell > 0$  does not depend on  $\epsilon \in (0, 1)$ .

**Step 4.** To proceed, we argue by induction. Taking the difference between the equation satisfied by  $v_{\epsilon,i+1}$  and the equation satisfied by  $v_{\epsilon,i}$  we get

$$-(\Delta_{g_z} + p\bar{u}_\epsilon)(v_{\epsilon,i+1} - v_{\epsilon,i}) = K_\epsilon(v_{\epsilon,i}) - K_\epsilon(v_{\epsilon,i-1}) + (\Delta_{\dot{g}} + D)(v_{\epsilon,i} - v_{\epsilon,i-1}),$$

and one proves by induction that

$$\|\nabla_{\dot{g}}^\ell(v_{\epsilon,i+1} - v_{\epsilon,i})\|_{C_\epsilon^{2,\alpha}(B_\epsilon(\Lambda))} \leq C_\ell \epsilon^{i+1-\frac{2}{p-1}},$$

where  $C_\ell > 0$  does not depend on  $\epsilon \in (0, 1)$ . The proof uses the arguments already employed in Step 2 and Step 3. There is though one additional argument which is needed to estimate the nonlinear term  $K_\epsilon(v)$ . The key observation is that  $v_{\epsilon,i}$  vanishes on  $\partial B_\epsilon(\Lambda)$  and if

$$\|\nabla_{g_z} v_{\epsilon,i}\|_{L^\infty(B_\epsilon(\Lambda))} \leq C\epsilon^{-\frac{2}{p-1}},$$

then, we have the pointwise estimate

$$(4.19) \quad |v_{\epsilon,i}| \leq \frac{1}{2}\bar{u}_\epsilon,$$

provided  $\epsilon$  is chosen close enough to 0, since by Hopf boundary Lemma  $U$  has non-zero normal derivative on  $T_\epsilon(\Lambda)$ . Hence, we are entitled to write

$$K_\epsilon(v_{\epsilon,i}) = \bar{u}_\epsilon^p ((1 + w_{\epsilon,i})^p - 1 - pw_{\epsilon,i}),$$

where

$$w_{\epsilon,i} := \frac{v_{\epsilon,i}}{\bar{u}_\epsilon},$$

takes values into  $[-1/2, 1/2]$ . In particular, we can use standard Taylor's expansion to evaluate the nonlinear term  $K_\epsilon$ . Details are left to the reader.  $\square$

Observe that (4.19) also implies that, if  $i \in \mathbb{N}$  is fixed,  $u_{\epsilon,i} > 0$  in  $B_\epsilon(\Lambda)$  provided  $\epsilon$  is chosen small enough.

5. ANALYSIS OF THE LINEARIZED OPERATOR ABOUT  $u_{\epsilon,i}$ 

We keep the notations of the previous section. In particular,  $u_{\epsilon,i}$  denotes one of the approximate solutions which have been defined in (4.15). In this section we are interested in the mapping properties of the linearized operator about  $u_{\epsilon,i}$ , namely

$$L_{\epsilon,i} := - \left( \Delta + p u_{\epsilon,i}^{p-1} \right).$$

We will exploit the fact that, in some sense to be made precise, this operator is close to the operator

$$\mathcal{L}_\epsilon := - \left( \Delta_{\bar{g}} + p \bar{u}_\epsilon^{p-1} \right),$$

whose eigenvalues are explicitly given by

$$(5.20) \quad \frac{\mu_i}{\epsilon^2} + \lambda_j,$$

where we recall that  $(\mu_j)_{j \geq 0}$  are the eigenvalues of  $L$  (defined in (4.10)) and where

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$$

are the eigenvalues of  $-\Delta_{\bar{g}}$  on  $\Lambda$ .

There are some remarks which are straightforward but nevertheless very important. All rely on the fact that, as already mentioned,  $\mu_0 < 0 < \mu_1$ .

- (i) The Morse index of  $\mathcal{L}_\epsilon$ , which is defined to be the maximal dimension of the subspaces of  $H_0^1(B_\epsilon(\Lambda))$  over which the quadratic form

$$(5.21) \quad \mathcal{Q}_\epsilon(v) := \int_{B_\epsilon(\Lambda)} (|\nabla_{\bar{g}} v|^2 - p \bar{u}_\epsilon^{p-1} v^2) \, \text{dvol}_{\bar{g}},$$

is definite negative, is a decreasing function of  $\epsilon$ . Observe that  $\mathcal{Q}_\epsilon$  is defined using the volume form associated to the metric  $\bar{g}$  on  $N\Lambda$  and

$$|\nabla_{\bar{g}} v|^2 = |\nabla_{\dot{g}} v|^2 + |\nabla_{g_z} v|^2.$$

By Weyl's formula (see for example [9]) it is known that the number of eigenvalues of  $-\Delta_{\bar{g}}$  (counted with multiplicity), which are less than  $\lambda > 0$  is asymptotic to  $\lambda^{k/2}$  as  $\lambda$  tends to infinity. Therefore, taking into account (5.20) (see also (ii) here below), we get an estimate of the Morse index of  $\mathcal{L}_\epsilon$ , namely

$$\text{Index } \mathcal{L}_\epsilon \sim \epsilon^{-k}.$$

- (ii) Observe that

$$\frac{\mu_i}{\epsilon^2} + \lambda_j \geq \frac{\mu_1}{\epsilon^2} > 0,$$

for all  $i \geq 1$  and all  $j \geq 0$ . In particular, the eigenfunctions associated to negative eigenvalues of  $\mathcal{L}_\epsilon$  are of the form

$$(y, z) \mapsto \phi_0(z/\epsilon) \psi(y),$$

where  $\psi$  is an eigenfunction of  $-\Delta_{\dot{g}}$  and where we recall that  $\phi_0$  is the eigenfunction of  $L$  associated to  $\mu_0$ .

- (iii) The eigenvalues of  $\mathcal{L}_\epsilon$  are monotone functions of  $\epsilon$  and in fact

$$\partial_\epsilon \left( \frac{\mu_0}{\epsilon^2} + \lambda_j \right) = -2 \frac{\mu_0}{\epsilon^3}.$$

In particular, the Morse index of  $\mathcal{L}_\epsilon$  is a decreasing function of  $\epsilon$ .

(iv) The spectrum of  $\mathcal{L}_\epsilon$  contains 0 if and only if

$$\epsilon = \sqrt{\frac{-\mu_0}{\lambda_j}},$$

and for this special values of  $\epsilon$ , the operator  $\mathcal{L}_\epsilon$  (under 0 Dirichlet boundary conditions) is not invertible.

Having these remarks in mind, we now explain the argument we would use if we were to work with the operator  $\mathcal{L}_\epsilon$  instead of  $L_{\epsilon,i}$ .

We define

$$\mathcal{Z}_\epsilon := \{\epsilon > 0 : \exists j \in \mathbb{N}, \quad \mu_0 + \epsilon^2 \lambda_j = 0\},$$

which corresponds to the set of  $\epsilon$ 's for which the operator

$$\begin{aligned} H_0^1(B_\epsilon(\Lambda)) \cap H^2(B_\epsilon(\Lambda)) &\rightarrow L^2(B_\epsilon(\Lambda)), \\ w &\mapsto \mathcal{L}_\epsilon w, \end{aligned}$$

is not invertible. Now, if  $\epsilon \notin \mathcal{Z}_\epsilon$ , we can estimate the norm of the inverse of  $\mathcal{L}_\epsilon$  by a constant times  $1/\delta_\epsilon$  where  $\delta_\epsilon$  is the distance from 0 to the spectrum of  $\mathcal{L}_\epsilon$ , namely

$$\delta_\epsilon := \min \left\{ \left| \frac{\mu_0}{\epsilon^2} + \lambda_j \right| : j \in \mathbb{N} \right\}.$$

We fix  $N \geq \max(2, k)$  and we define, for all  $\epsilon$  such that  $0 < \epsilon < 1$ , the set

$$S_{\epsilon,N} := \{\bar{\epsilon} \in (\epsilon, 2\epsilon) : (\bar{\epsilon} - \epsilon^N, \bar{\epsilon} + \epsilon^N) \cap \mathcal{Z}_\epsilon = \emptyset\}.$$

Property (i), which makes use of Weyl's asymptotic formula, implies in particular that  $\epsilon - \text{meas}(S_{\epsilon,N})$  cannot be larger than a constant times  $\epsilon^{N-k}$  and, for any  $\bar{\epsilon} \in S_{\epsilon,N}$  we know from property (iii) (or from direct estimate) that the norm of the inverse of  $\mathcal{L}_{\bar{\epsilon}}$  (defined as above) is bounded by a constant times  $\epsilon^{3-N}$ .

Therefore, if

$$S_N := \bigcup_{\epsilon \in (0,1)} S_{\epsilon,N},$$

then, for all  $\epsilon \in S_N$ , the norm of the inverse of  $\mathcal{L}_\epsilon$  from  $L^2(B_\epsilon(\Lambda))$  into  $L^2(B_\epsilon(\Lambda))$  is controlled by a constant times  $\epsilon^{3-N}$ . Moreover, if  $N - k \geq 2$  we have

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^\alpha} (\epsilon - \text{meas}(S_N \cap (0, \epsilon))) = 0,$$

provided  $\alpha \in (1, N - k)$ .

This is the argument we will try to adapt to the operator  $L_{\epsilon,i}$ . The main difficulty is that we will not be able to use separation of variables anymore, instead we will use the fact that the operators  $L_{\epsilon,i}$  and  $\mathcal{L}_\epsilon$  are close.

**5.1. Estimating the Morse index of  $L_{\epsilon,i}$ .** In this section, given  $i \geq 0$ , we recover partially Property (i) for the operator  $L_{\epsilon,i}$ . We define the quadratic form associated to  $L_{\epsilon,i}$  by

$$(5.22) \quad \mathcal{Q}_{\epsilon,i}(v) := \int_{B_\epsilon(\Lambda)} \left( |\nabla_{g_o} v|^2 - p u_{\epsilon,i}^{p-1} v^2 \right) \text{dvol}_{g_o}.$$

Observe that the volume form is the one associated to the Euclidean metric  $g_o$ . Similarly, the norm of the gradient of the function  $v$  is computed using the Euclidean metric  $g_o$ . Comparing  $\mathcal{Q}_{\epsilon,i}$  to the quadratic form  $\mathcal{Q}_\epsilon$  which was defined in (5.21), we prove the

**Lemma 5.1.** *Assume that  $i \in \mathbb{N}$  is fixed. Then, there exists a constant  $C > 0$  such that*

$$\text{Index } L_{\epsilon,i} \leq C\epsilon^{-k},$$

for all  $\epsilon > 0$  close enough to 0.

*Proof.* Let  $v \in H_0^1(B_\epsilon(\Lambda))$  which is normalized so that  $\|v\|_{L^2(B_\epsilon(\Lambda))} = 1$  and which satisfies  $\mathcal{Q}_{\epsilon,i}(v) \leq 0$ . We want to estimate  $\mathcal{Q}_\epsilon(v)$ . Observe that the difference between  $\mathcal{Q}_\epsilon$  and  $\mathcal{Q}_{\epsilon,i}$  can be attributed to three different phenomena. First the difference between the square of the norm of the gradient of  $v$  when the Euclidean metric or the product metric  $\bar{g}$  are used, second the difference between the potentials  $\bar{u}_\epsilon$  and  $u_{\epsilon,i}$  and finally the difference between the volume forms when the Euclidean metric or the product metric  $\bar{g}$  are used.

Using the result of Lemma 3.1, it is easy to check that

$$||\nabla_{g_o} v|^2 - |\nabla_{\bar{g}} v|^2| \leq C\epsilon |\nabla_{g_o} v|^2.$$

Similarly, it follows from Proposition 4.1, that

$$|\bar{u}_\epsilon^{p-1} - u_{\epsilon,i}^{p-1}| \leq C\epsilon u_{\epsilon,i}^{p-1}.$$

Finally, it follows once more from Lemma 3.1 that the difference between the volume forms can be estimated in local coordinates by

$$|\sqrt{\det \bar{g}} - \sqrt{\det g_o}| \leq C\epsilon \sqrt{\det \bar{g}}.$$

Since  $\mathcal{Q}_{\epsilon,i}(v) \leq 0$ , we find that

$$\begin{aligned} \mathcal{Q}_\epsilon(v) &= \mathcal{Q}_\epsilon(v) - \mathcal{Q}_{\epsilon,i}(v) + \mathcal{Q}_{\epsilon,i}(v) \\ &\leq \mathcal{Q}_\epsilon(v) - \mathcal{Q}_{\epsilon,i}(v) \\ &\leq C\epsilon \int_{B_\epsilon(\Lambda)} (|\nabla_{g_o} v|^2 + u_{\epsilon,i}^{p-1} v^2) \, \text{dvol}_{g_o}. \end{aligned}$$

Moreover,  $\mathcal{Q}_{\epsilon,i}(v) \leq 0$  also implies that

$$\int_{B_\epsilon(\Lambda)} |\nabla_{g_o} v|^2 \, \text{dvol}_{g_o} \leq \int_{B_\epsilon(\Lambda)} p u_{\epsilon,i}^{p-1} v^2 \, \text{dvol}_{g_o} \leq C\epsilon^{-2} \int_{B_\epsilon(\Lambda)} v^2 \, \text{dvol}_{\bar{g}}.$$

Therefore, we have

$$\mathcal{Q}_\epsilon(v) \leq \frac{C}{\epsilon} \int_{B_\epsilon(\Lambda)} v^2 \, \text{dvol}_{\bar{g}} = \frac{C}{\epsilon}.$$

Using these, we see that the index of  $L_{\epsilon,i}$  is bounded by the dimension of the space spanned by the eigenfunctions of  $\mathcal{L}_\epsilon$  associated to eigenvalues less than or equal to  $C/\epsilon$ . Using Weyl's asymptotic formula and the explicit expression for the eigenvalues of  $\mathcal{L}_\epsilon$ , we conclude that the index of  $L_{\epsilon,i}$  is bounded by a constant times  $\epsilon^{-k}$  and this completes the proof of the result.  $\square$

**5.2. Decomposition of eigenfunctions associated to small eigenvalues.** Given  $i \geq 0$ , in this section, we recover Property (ii) for an operator close to  $L_{\epsilon,i}$ . Using the function  $a$  defined in (3.9), we set

$$\tilde{L}_{\epsilon,i} := a L_{\epsilon,i}.$$

Observe that  $L_{\epsilon,i}$  is self-adjoint with respect to  $L^2(B_\epsilon(\Lambda), g_o)$  while  $\tilde{L}_{\epsilon,i}$  is self-adjoint with respect to  $L^2(B_\epsilon(\Lambda), \bar{g})$ . Indeed, we have

$$\begin{aligned} \int_{B_\epsilon(\Lambda)} v \tilde{L}_{\epsilon,i} w \, d\text{vol}_{\bar{g}} &= \int_{B_\epsilon(\Lambda)} v L_{\epsilon,i} w \, d\text{vol}_{g_o} \\ &= \int_{B_\epsilon(\Lambda)} w L_{\epsilon,i} v \, d\text{vol}_{g_o} \\ &= \int_{B_\epsilon(\Lambda)} w \tilde{L}_{\epsilon,i} v \, d\text{vol}_{\bar{g}}. \end{aligned}$$

Also observe that the Morse index of  $\tilde{L}_{\epsilon,i}$  is equal to the Morse index of  $L_{\epsilon,i}$  since the two associated quadratic forms are equal.

Recall that we have denoted by  $\phi_0$  the eigenfunction of  $-(\Delta + p\bar{U}^{p-1})$  associated to the eigenvalue  $\mu_0 < 0$  which is normalized to have  $L^2$  norm equal to 1. Observe that  $\phi_0$  is radial and hence we can define

$$\phi_{0,\epsilon}(y, z) := \phi_0(|z|/\epsilon),$$

on  $B_\epsilon(\Lambda)$ .

Let  $v$  be an eigenfunction of  $\tilde{L}_{\epsilon,i}$  associated to the eigenvalue  $\nu$ . Hence

$$\tilde{L}_{\epsilon,i} v = \nu v,$$

in  $B_\epsilon(\Lambda)$  and  $v = 0$  on  $T_\epsilon(\Lambda)$ . We decompose

$$(5.23) \quad v(y, z) = \phi_{0,\epsilon}(z) \psi(y) + \bar{v}(y, z),$$

where  $\psi$  is a function defined on  $\Lambda$  and

$$\int_{B_\epsilon(\Lambda)} \bar{v} \phi_{0,\epsilon} h \, d\text{vol}_{\bar{g}} = 0,$$

for any  $h \in L^2(\Lambda)$ . Observe that the orthogonality condition is expressed using the metric  $\bar{g}$  and not  $g_o$ . As usual, we identify  $B_\epsilon(\Lambda)$  with a tubular neighborhood of the zero section in  $N\Lambda$ . We have the :

**Lemma 5.2.** *There exists constants  $C_0, C > 0$  such that, if  $v$  is a solution of  $\tilde{L}_{\epsilon,i} v = \nu v$  which is decomposed as in (5.23) and if we further assume that*

$$\nu \leq \frac{C_0}{\epsilon^2},$$

then

$$(5.24) \quad \int_{B_\epsilon(\Lambda)} \left( |\nabla_{\bar{g}} \bar{v}|^2 + \frac{1}{\epsilon^2} \bar{v}^2 \right) d\text{vol}_{\bar{g}} \leq \frac{C}{\epsilon} \int_{B_\epsilon(\Lambda)} v^2 d\text{vol}_{\bar{g}}.$$

*Proof.* For notational convenience, we set  $v_0(y, z) = \phi_{0,\epsilon}(z) \psi(y)$  so that  $v = v_0 + \bar{v}$ . In the proof, one has to be careful since there are two different metrics which are used in  $B_\epsilon(\Lambda)$ . The first metric is the Euclidean metric  $g_o$  with respect to which  $L_{\epsilon,i}$  is self-adjoint and the second metric is  $\bar{g}$  with respect to which  $\mathcal{L}_\epsilon$  is self-adjoint.

**Step 1.** We exploit the fact that  $\tilde{L}_{\epsilon,i} v = \nu v$  by multiplying this equation by  $v$  and integrating the result over  $B_\epsilon(\Lambda)$  to find

$$(5.25) \quad \int_{B_\epsilon(\Lambda)} \left( |\nabla_{g_o} v|^2 - p u_{\epsilon,i}^{p-1} v^2 \right) d\text{vol}_{g_o} = \nu \int_{B_\epsilon(\Lambda)} v^2 d\text{vol}_{\bar{g}}.$$

Since  $u_{\epsilon,i}^{p-1} \leq C\epsilon^{-2}$  and since the volume forms associated to  $g_o$  and  $\bar{g}$  are equivalent in a neighborhood of  $\Lambda$ , we get the estimate

$$(5.26) \quad \int_{B_\epsilon(\Lambda)} |\nabla_{g_o} v|^2 d\text{vol}_{g_o} \leq \left( \nu + \frac{C}{\epsilon^2} \right) \int_{B_\epsilon(\Lambda)} v^2 d\text{vol}_{\bar{g}}.$$

**Step 2.** Next, we exploit the fact that the eigenvalues of  $\mathcal{L}_\epsilon$  are explicitly known and the orthogonal decomposition of  $v$  implies that

$$\frac{\mu_1}{\epsilon^2} \int_{B_\epsilon(\Lambda)} \bar{v}^2 d\text{vol}_{\bar{g}} \leq \int_{B_\epsilon(\Lambda)} (|\nabla_{\bar{g}} \bar{v}|^2 - p\bar{u}_\epsilon^{p-1} \bar{v}^2) d\text{vol}_{\bar{g}}.$$

Using the fact that  $p\bar{u}_\epsilon^{p-1} \leq C\epsilon^{-2}$ , we conclude that there exists a constant  $C_1 > 0$  such that

$$C_1 \int_{B_\epsilon(\Lambda)} \left( |\nabla_{\bar{g}} \bar{v}|^2 + \frac{1}{\epsilon^2} \bar{v}^2 \right) d\text{vol}_{\bar{g}} \leq \int_{B_\epsilon(\Lambda)} (|\nabla_{\bar{g}} \bar{v}|^2 - p\bar{u}_\epsilon^{p-1} \bar{v}^2) d\text{vol}_{\bar{g}}.$$

Since  $\mathcal{L}_\epsilon v_0$  is  $L^2(B_\epsilon(\Lambda), \bar{g})$  orthogonal to  $\bar{v}$ , we conclude that

$$C_1 \int_{B_\epsilon(\Lambda)} \left( |\nabla_{\bar{g}} \bar{v}|^2 + \frac{1}{\epsilon^2} \bar{v}^2 \right) d\text{vol}_{\bar{g}} \leq \int_{B_\epsilon(\Lambda)} (\nabla_{\bar{g}} v \cdot \nabla_{\bar{g}} \bar{v} - p\bar{u}_\epsilon^{p-1} v\bar{v}) d\text{vol}_{\bar{g}}.$$

**Step 3.** As in the proof of Lemma 5.1, we can replace the metric  $\bar{g}$  by the metric  $g_o$  and the function  $\bar{u}_\epsilon$  by  $u_{\epsilon,i}$  on the right hand side and, using the results of Lemma 3.1 and Proposition 4.1, we conclude that

$$\begin{aligned} C_1 \int_{B_\epsilon(\Lambda)} \left( |\nabla_{\bar{g}} \bar{v}|^2 + \frac{1}{\epsilon^2} \bar{v}^2 \right) d\text{vol}_{\bar{g}} &\leq \int_{B_\epsilon(\Lambda)} (\nabla_{g_o} v \cdot \nabla_{g_o} \bar{v} - p u_{\epsilon,i}^{p-1} v\bar{v}) d\text{vol}_{g_o} \\ &+ C\epsilon \int_{B_\epsilon(\Lambda)} (|\nabla_{\bar{g}} \bar{v}|^2 + |\nabla_{\bar{g}} v|^2) d\text{vol}_{\bar{g}} \\ &+ \frac{C}{\epsilon} \int_{B_\epsilon(\Lambda)} (\bar{v}^2 + v^2) d\text{vol}_{\bar{g}}. \end{aligned}$$

Since  $\tilde{L}_{\epsilon,i} v = \nu v$ , we conclude that

$$\begin{aligned} C_1 \int_{B_\epsilon(\Lambda)} \left( |\nabla_{\bar{g}} \bar{v}|^2 + \frac{1}{\epsilon^2} \bar{v}^2 \right) d\text{vol}_{\bar{g}} &\leq \nu \int_{B_\epsilon(\Lambda)} \bar{v}^2 d\text{vol}_{\bar{g}} \\ &+ C\epsilon \int_{B_\epsilon(\Lambda)} (|\nabla_{\bar{g}} \bar{v}|^2 + |\nabla_{\bar{g}} v|^2) d\text{vol}_{\bar{g}} \\ &+ \frac{C}{\epsilon} \int_{B_\epsilon(\Lambda)} (\bar{v}^2 + v^2) d\text{vol}_{\bar{g}}. \end{aligned}$$

On the right hand side, the terms in  $\bar{v}$  can be absorbed in the left hand side provided  $\epsilon$  is chosen small enough and  $\nu \leq C_1/(2\epsilon^2)$ . We conclude that

$$\int_{B_\epsilon(\Lambda)} \left( |\nabla_{\bar{g}} \bar{v}|^2 + \frac{1}{\epsilon^2} \bar{v}^2 \right) d\text{vol}_{\bar{g}} \leq C\epsilon \int_{B_\epsilon(\Lambda)} \left( |\nabla_{\bar{g}} v|^2 + \frac{1}{\epsilon^2} v^2 \right) d\text{vol}_{\bar{g}}.$$

This together with (5.26) implies that that

$$\int_{B_\epsilon(\Lambda)} \left( |\nabla_{\bar{g}} \bar{v}|^2 + \frac{1}{\epsilon^2} \bar{v}^2 \right) d\text{vol}_{\bar{g}} \leq \frac{C}{\epsilon} \int_{B_\epsilon(\Lambda)} v^2 d\text{vol}_{\bar{g}},$$

and this completes the proof of the lemma.  $\square$

**5.3. Exploiting Kato's result.** In this section, we estimate the rate of change of eigenvalues of  $\tilde{L}_{\epsilon,i}$  as  $\epsilon$  varies. In other words, we obtain for the operator  $\tilde{L}_{\epsilon,i}$  a result, which is close to the Property (iii) which was straightforward for the operator  $\mathcal{L}_\epsilon$ .

Let us explain the proof in the case where  $\nu := \nu(\epsilon)$  is a simple eigenvalue for  $\tilde{L}_{\epsilon,i}$ . It is known that in this case  $\nu$  depends smoothly on  $\epsilon$ . To proceed, we need to work with functions defined on a fixed domain which does not depend on  $\epsilon$ . Hence, we parameterize  $B_\epsilon(\Lambda)$  using

$$\begin{aligned} F_\epsilon : \Omega_1(\Lambda) &\rightarrow B_\epsilon(\Lambda) \\ (y, z) &\mapsto y + \epsilon z, \end{aligned}$$

where  $\Omega_1(\Lambda) := \{(y, z) \in N\Lambda : |z| < 1\}$ . Observe that

$$F_\epsilon^* \bar{g} = \dot{g} + \epsilon^2 dz^2.$$

We define the operator  $\hat{L}_{\epsilon,i}$  by

$$\hat{L}_{\epsilon,i}(v \circ F_\epsilon) = (\tilde{L}_{\epsilon,i} v) \circ F_\epsilon.$$

Let  $v := v(\epsilon)$  be the eigenfunction of  $\tilde{L}_{\epsilon,i}$  associated to  $\nu = \nu(\epsilon)$ . By definition of  $\hat{L}_{\epsilon,i}$ , we have

$$(5.27) \quad \hat{L}_{\epsilon,i} w = \nu w,$$

where  $w := v \circ F_\epsilon$ . Without loss of generality we can assume that  $w$  depends smoothly on  $\epsilon$  and is normalized so that

$$\int_{\Omega_1(\Lambda)} w^2 d\text{vol}_{\bar{g}} = 1.$$

Differentiation of (5.27) with respect to  $\epsilon$ , yields

$$\hat{L}_{\epsilon,i}(\partial_\epsilon w) + (\partial_\epsilon \hat{L}_{\epsilon,i}) w = (\partial_\epsilon \nu) w + \nu (\partial_\epsilon w).$$

Multiplying this equation by  $w$  and integrating over  $\Omega_1(\Lambda)$ , we get

$$\partial_\epsilon \nu = \int_{\Omega_1(\Lambda)} w (\partial_\epsilon \hat{L}_{\epsilon,i}) w d\text{vol}_{\bar{g}}.$$

When the eigenspaces are not simple, we can interpret  $\partial_\epsilon \nu$  as a set-valued function which takes into account the possibility that the eigenvalue splits into a number of separate eigenvalues (see [10] and [5]). The estimate for the elements of this set of derivatives is given by

$$\partial_\epsilon \nu \in \left\{ \int_{\Omega_1(\Lambda)} w (\partial_\epsilon \hat{L}_{\epsilon,i}) w d\text{vol}_{\bar{g}} : \hat{L}_{\epsilon,i} w = \nu w \quad \text{and} \quad \int_{\Omega_1(\Lambda)} w^2 d\text{vol}_{\bar{g}} = 1 \right\}.$$

We have the :

**Lemma 5.3.** *There exists a constant  $C_2 > 0$  such that if  $i \in \mathbb{N}$  is fixed and if  $\nu$  is an eigenvalue of  $\tilde{L}_{\epsilon,i}$  such that*

$$\nu \leq \frac{C_1}{\epsilon^2},$$

*where  $C_1$  is the constant defined in Lemma 5.2, then*

$$\partial_\epsilon \nu \geq \frac{C_2}{\epsilon^3},$$



for all  $\epsilon$  small enough.

*Proof.* We use the decomposition of the Laplacian which was given in Lemma 3.2 together with the estimates (4.17) for  $v_{\epsilon,i}$  and  $\partial_{\epsilon} v_{\epsilon,i}$  which were given in Proposition 4.1. With little work we conclude that

$$\begin{aligned} \int_{\Omega_1(\Lambda)} w (\partial_{\epsilon} \tilde{L}_{\epsilon,i}) w \, d\text{vol}_{\tilde{g}} &\geq -\frac{2}{\epsilon^3} \int_{\Omega_1(\Lambda)} (|\nabla_{g_z} w|^2 - p U^{p-1} w^2) \, d\text{vol}_{\tilde{g}} \\ &\quad - C \int_{\Omega_1(\Lambda)} \left( |\nabla_{\tilde{g}} w|^2 + \frac{1}{\epsilon^2} (|\nabla_{g_z} w|^2 + w^2) \right) d\text{vol}_{\tilde{g}}. \end{aligned}$$

Indeed, the expression of  $\tilde{L}_{\epsilon,i}$  in local coordinates  $t := (t_1, \dots, t_k)$  on  $\Lambda$  and  $z := (z_1, \dots, z_n)$  on the normal section can be written as

$$\tilde{L}_{\epsilon,i} = -\tilde{a} \left( \Delta_{\tilde{g}} + \frac{1}{\epsilon^2} \left( \Delta_{g_z} + p(U + \epsilon^{\frac{2}{p-1}} \tilde{v}_{\epsilon,i})^{p-1} \right) \right) + \epsilon \tilde{D}^{(2)} + \tilde{D}^{(1)},$$

where  $\tilde{a} := a \circ F_{\epsilon}$ ,  $\tilde{v}_{\epsilon,i} := v_{\epsilon,i} \circ F_{\epsilon}$  and where  $\tilde{D}^{(2)}$  (respectively  $\tilde{D}^{(1)}$ ) is a second order (respectively first order) partial differential operator in  $\partial_{t_j}$  and  $\epsilon^{-1} \partial_{z_i}$  with smooth coefficients in  $\Omega_1(\Lambda)$ .

Differentiating with respect to  $\epsilon$  and using (4.17) we conclude that

$$\partial_{\epsilon} \tilde{L}_{\epsilon,i} = \frac{2}{\epsilon^3} (\Delta_{g_z} + p U^{p-1}) + \hat{D}^{(2)} + \hat{D}^{(1)} + \frac{1}{\epsilon^2} \hat{D}^{(0)},$$

where  $\hat{D}^{(j)}$  is a  $j$ -th order partial differential operator in  $\partial_{t_j}$  and  $\epsilon^{-2} \partial_{z_i}$  with smooth coefficients in  $\Omega_1(\Lambda)$ .

Now, we decompose the eigenfunctions  $w$  satisfying  $\hat{L}_{\epsilon,i} w = \nu w$  into  $w = w_0 + \bar{w}$  as in Lemma 5.2 to get

$$\begin{aligned} \int_{\Omega_1(\Lambda)} w (\partial_{\epsilon} \tilde{L}_{\epsilon,i}) w \, d\text{vol}_{\tilde{g}} &\geq -\frac{2}{\epsilon^3} \int_{\Omega_1(\Lambda)} (|\nabla_{g_z} w_0|^2 - p U^{p-1} w_0^2) \, d\text{vol}_{\tilde{g}} \\ &\quad - \frac{2}{\epsilon^3} \int_{\Omega_1(\Lambda)} (|\nabla_{g_z} \bar{w}|^2 - p U^{p-1} \bar{w}^2) \, d\text{vol}_{\tilde{g}} \\ &\quad - C \int_{\Omega_1(\Lambda)} \left( |\nabla_{\tilde{g}} w|^2 + \frac{1}{\epsilon^2} (|\nabla_{g_z} w|^2 + w^2) \right) d\text{vol}_{\tilde{g}}. \end{aligned}$$

Since

$$\int_{\Omega_1(\Lambda)} (|\nabla_{g_z} w_0|^2 - p U^{p-1} w_0^2) \, d\text{vol}_{\tilde{g}} = \mu_0 \int_{\Omega_1(\Lambda)} w_0^2 \, d\text{vol}_{\tilde{g}},$$

we conclude, using the estimate (5.24) and (5.25) in the proof of Lemma 5.2 that

$$\int_{\Omega_1(\Lambda)} w (\partial_{\epsilon} \tilde{L}_{\epsilon,i}) w \, d\text{vol}_{\tilde{g}} \geq -\frac{2\mu_0}{\epsilon^3} \int_{\Omega_1(\Lambda)} w_0^2 \, d\text{vol}_{\tilde{g}} - \frac{C}{\epsilon^2} \int_{\Omega_1(\Lambda)} w^2 \, d\text{vol}_{\tilde{g}}.$$

Since

$$\int_{\Omega_1(\Lambda)} w_0^2 \, d\text{vol}_{\tilde{g}} = \int_{\Omega_1(\Lambda)} w^2 \, d\text{vol}_{\tilde{g}} - \int_{\Omega_1(\Lambda)} \bar{w}^2 \, d\text{vol}_{\tilde{g}},$$

we can again use the result of Lemma 5.2 to conclude that

$$\int_{\Omega_1(\Lambda)} w (\partial_{\epsilon} \tilde{L}_{\epsilon,i}) w \, d\text{vol}_{\tilde{g}} \geq -\left( \frac{2\mu_0}{\epsilon^3} + \frac{C}{\epsilon^2} \right) \int_{\Omega_1(\Lambda)} w^2 \, d\text{vol}_{\tilde{g}}.$$

One can then choose the constant  $C_2 > 0$  to be any number  $C_2 < -2\mu_0$ .  $\square$

6. UNIFORM INVERTIBILITY OF  $L_{\epsilon,i}$ 

We now have all the ingredients to apply to the operator  $L_{\epsilon,i}$ , the strategy which was outlined at the beginning of §5 for the operator  $\mathcal{L}_\epsilon$ .

We fix  $i \geq 0$  and  $\epsilon > 0$ . We denote by  $\Sigma_{\epsilon,i}$  the spectrum of  $\tilde{L}_{\epsilon,i}$  and we define

$$\mathcal{Z}_{\epsilon,i} := \{\epsilon > 0 : 0 \in \Sigma_{\epsilon,i}\},$$

which corresponds to the set of  $\epsilon$ 's for which the operator

$$H_0^1(B_\epsilon(\Lambda)) \cap H^2(B_\epsilon(\Lambda)) \rightarrow L^2(B_\epsilon(\Lambda)),$$

$$w \mapsto \tilde{L}_{\epsilon,i} w,$$

is not invertible. It is standard that, if  $\epsilon \notin \mathcal{Z}_{\epsilon,i}$ , one can estimate the norm of the inverse of  $\tilde{L}_{\epsilon,i}$  by a constant times  $1/\delta_{\epsilon,i}$  where  $\delta_{\epsilon,i}$  is the distance from 0 to the spectrum of  $\tilde{L}_{\epsilon,i}$ , namely

$$\delta_{\epsilon,i} := \min \{|\nu| : \nu \in \Sigma_{\epsilon,i}\}.$$

We fix  $N \geq \max(2, k)$  and we define, for all  $\epsilon$  such that  $0 < \epsilon < 1$ , the set

$$S_{\epsilon,i,N} := \{\bar{\epsilon} \in (\epsilon, 2\epsilon) : (\bar{\epsilon} - \epsilon^N, \bar{\epsilon} + \epsilon^N) \cap \mathcal{Z}_{\epsilon,i} = \emptyset\}.$$

The result of Lemma 5.1, implies that  $\epsilon - \text{meas}(S_{\epsilon,i,N})$  cannot be larger than a constant times  $\epsilon^{N-k}$  and, for any  $\bar{\epsilon} \in S_{\epsilon,i,N}$  we know from Lemma 5.3 that the norm of the inverse of  $\tilde{L}_{\epsilon,i}$  (defined as above) is bounded by a constant times  $\epsilon^{3-N}$  and, since  $\tilde{L}_{\epsilon,i} = a L_{\epsilon,i}$  where  $a$  is bounded away from 0, a similar property holds for  $L_{\epsilon,i}$ .

Therefore, if

$$S_{i,N} := \bigcup_{\epsilon \in (0,1)} S_{\epsilon,i,N},$$

then, for all  $\epsilon \in S_{i,N}$ , the norm of the inverse of  $L_{\epsilon,i}$  defined from  $L^2(B_\epsilon(\Lambda))$  into  $L^2(B_\epsilon(\Lambda))$ , is controlled by a constant times  $\epsilon^{3-N}$ . Moreover, if  $N - k \geq 2$  we have

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^\alpha} (\epsilon - \text{meas}(S_{i,N} \cap (0, \epsilon))) = 0,$$

provided  $\alpha < N - k$ .

**Definition 6.1.** We define  $C_0^1(B_\epsilon(\Lambda))$  to be the subspace of  $C^1(B_\epsilon(\Lambda))$  spanned by functions which vanish on  $T_\epsilon(\Lambda)$ .

Now it is enough to invoke Schauder's estimates to estimate the norm of the inverse of  $L_{\epsilon,i}$  when defined from  $C^0(B_\epsilon(\Lambda))$  into  $C_0^1(B_\epsilon(\Lambda))$  (here we use the Euclidean metric to estimate the norm of the partial derivatives of functions). Using Schauder's estimates to control the norm of the inverse of  $L_{\epsilon,i}$  between  $C^\ell$  spaces starting from the knowledge of its norm between Lebesgue spaces, we loose a few powers of  $\epsilon$ , say  $\epsilon^{-N_0}$ , where  $N_0$  only depends on the dimension  $m$ . We have proven the :

**Lemma 6.1.** Given  $i \geq 0$  and  $N \geq k + 2$ , there exist  $S_{i,N} \subset (0, +\infty)$  and  $N_0 \in \mathbb{N}$  such that, for all  $\epsilon \in S_{i,N}$  the operator  $L_{\epsilon,i}$  is invertible and the norm of its inverse defined from  $C^0(B_\epsilon(\Lambda))$  into  $C_0^1(B_\epsilon(\Lambda))$  is bounded by a constant times  $\epsilon^{3-N-N_0}$ . Moreover

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^\alpha} (\epsilon - \text{meas}(S_{i,N} \cap (0, \epsilon))) = 0,$$

provided  $\alpha < N - k$ .

## 7. A PERTURBATION ARGUMENT AND THE PROOF OF THEOREM 1.1

Thanks to the previous analysis, we can now give the proof of Theorem 1.1. We keep the notations introduced in the previous section.

As already mentioned, we perturb the approximate solution  $u_{\epsilon,i}$ . Therefore, we look for a solution  $u = u_{\epsilon,i} + v$  so that the equation to solve can be written as

$$(7.28) \quad L_{\epsilon,i}v = E_{\epsilon,i} + K_{\epsilon,i}(v),$$

where by definition

$$E_{\epsilon,i} := \Delta u_{\epsilon,i} + u_{\epsilon,i}^p,$$

and

$$K_{\epsilon,i}(v) := |u_{\epsilon,i} + v|^p - u_{\epsilon,i}^p - p u_{\epsilon,i}^{p-1} v.$$

As in the proof of Proposition 4.1, it will be convenient to observe that

$$K_{\epsilon,i}(v) := u_{\epsilon,i}^p \left( \left| 1 + \frac{w}{u_{\epsilon,i}} \right|^p - 1 - p \frac{w}{u_{\epsilon,i}} \right),$$

so that one can use Taylor's expansion to evaluate the nonlinear terms, provided  $w/u_{\epsilon,i}$  is small enough.

We fix  $\alpha > 1$  as in the statement of Theorem 1.1 and  $N \geq k + 2$ . Then, we choose  $i > 2(N + N_0) - 3$  and  $M$  such that

$$(7.29) \quad i + 2 - N - N_0 - \frac{2}{p-1} > M,$$

and

$$(7.30) \quad M > N + N_0 - 1 - \frac{2}{p-1}.$$

According to (4.16), we have

$$\|E_{\epsilon,i}\|_{L^\infty(B_\epsilon(\Lambda))} \leq C \epsilon^{i - \frac{p+1}{p-1}},$$

and we can use the result of Lemma 6.1 to evaluate the norm of  $L_{\epsilon,i}^{-1}$ , the inverse of  $L_{\epsilon,i}$ , by

$$\|L_{\epsilon,i}^{-1}\|_{C^0 \rightarrow C^1} \leq C \epsilon^{3-N-N_0}.$$

Recall that  $C_0^1(B_\epsilon(\Lambda))$  denotes the subspace of  $C^1(B_\epsilon(\Lambda))$  spanned by functions which vanish on  $T_\epsilon(\Lambda)$ . Now, assume that  $v \in C_0^1(B_\epsilon(\Lambda))$  satisfies  $\|v\|_{C_0^1(B_\epsilon(\Lambda))} \leq \epsilon^M$  where  $M$  is fixed as above. Since  $v$  vanished on  $T_\epsilon(\Lambda)$  and since the gradient of  $v$  is bounded by  $\epsilon^M$  we conclude that  $|v/u_{\epsilon,i}| \leq 1/2$  for all  $\epsilon > 0$  small enough. Hence we find

$$\|K_{\epsilon,i}(v_2) - K_{\epsilon,i}(v_1)\|_{L^\infty(B_\epsilon(\Lambda))} \leq C \epsilon^{M-2\frac{p-2}{p-1}} \|v_2 - v_1\|_{L^\infty(B_\epsilon(\Lambda))},$$

for all  $v_2, v_1 \in C_0^1(B_\epsilon(\Lambda))$  such that  $\|v_i\|_{C_0^1(B_\epsilon(\Lambda))} \leq \epsilon^M$ .

We can then rephrase the solvability of (1.2) as a fixed point problem

$$v := L_{\epsilon,i}^{-1}(E_{\epsilon,i} + K_{\epsilon,i}(v)),$$

and apply a standard fixed point argument for contraction mapping in

$$\{v \in C_0^1(B_\epsilon(\Lambda)) \quad : \quad \|v\|_{C^1(B_\epsilon(\Lambda))} \leq \epsilon^M\}.$$

The choice of  $M$  implies that we have a contraction mapping (this fact uses (7.30)) from this set into itself (this fact uses (7.29)). This completes the proof of Theorem 1.1.

## 8. A POHOZAEV TYPE ARGUMENT AND THE PROOF OF THEOREM 1.2

In this last section, we give a proof of Theorem 1.2 using a refined version of the celebrated technique introduced in [17] are usually referred to as *Pohozaev identity*. We exploit an appropriate use of test functions, and explicit computations carried out in Fermi coordinates.

We start with a general result in the following :

**Lemma 8.1.** *Assume  $D \subseteq \mathbb{R}^m$  is an open set, and let  $\phi \in C^2(D)$ , then*

$$\begin{aligned} \operatorname{div} \left( (\nabla u \cdot \nabla \phi) \nabla u - \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} u^{p+1} \right) \nabla \phi + \frac{1}{p+1} u \Delta \phi \nabla u \right) \\ + \left( \frac{1}{n} \Delta \phi |\nabla u|^2 - \nabla^2 \phi (\nabla u, \nabla u) \right) + \left( \frac{n-2}{2n} - \frac{1}{p+1} \right) |\nabla u|^2 \Delta \phi \\ - \frac{1}{p+1} u \nabla u \cdot \nabla \Delta \phi = 0, \end{aligned}$$

in  $D$ , provided  $u$  is a classical solution of  $\Delta u + u^p = 0$  in  $D$ .

*Proof.* Multiplying the equation  $\Delta u + u^p = 0$  by  $u$ , we get

$$(8.31) \quad \operatorname{div} (u \nabla u) = |\nabla u|^2 - u^{p+1}.$$

Next, we multiply the equation  $\Delta u + u^p = 0$  by  $\nabla \phi \cdot \nabla u$  to get after some simple manipulation

$$\begin{aligned} \operatorname{div} \left( (\nabla u \cdot \nabla \phi) \nabla u + \frac{1}{p+1} u^{p+1} \nabla \phi \right) - \nabla^2 \phi (\nabla u, \nabla u) - \frac{1}{p+1} u^{p+1} \Delta \phi \\ - \frac{1}{2} \nabla \phi \cdot \nabla (|\nabla u|^2) = 0. \end{aligned}$$

Trying to write the last term on the left hand side as a divergence and correcting, we conclude that

$$\begin{aligned} (8.32) \quad \operatorname{div} \left( (\nabla u \cdot \nabla \phi) \nabla u - \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} u^{p+1} \right) \nabla \phi \right) \\ - \nabla^2 \phi (\nabla u, \nabla u) + \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} u^{p+1} \right) \Delta \phi = 0. \end{aligned}$$

The result follows immediately from the use of (8.31) to eliminate the terms in  $u^{p+1}$  in (8.32).  $\square$

The previous result, together with the divergence theorem implies the following identity :

**Lemma 8.2.** *Assume that we are given a function  $\phi$  (at least  $C^2$ ) and  $u$  a solution of  $\Delta u + u^p = 0$  both defined on a bounded smooth domain  $\Omega$ . Further assume that  $u = 0$  on  $\partial\Omega$ , then*

$$\begin{aligned} \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \nabla \phi \cdot \nu \, d\sigma_{\partial\Omega} + \int_{\Omega} \left( \frac{1}{n} \Delta \phi |\nabla u|^2 - \nabla^2 \phi (\nabla u, \nabla u) \right) \, d\operatorname{vol}_{g_\phi} \\ + \left( \frac{n-2}{2n} - \frac{1}{p+1} \right) \int_{\Omega} |\nabla u|^2 \Delta \phi \, d\operatorname{vol}_{g_\phi} - \frac{1}{p+1} \int_{\Omega} u \nabla u \cdot \nabla \Delta \phi \, d\operatorname{vol}_{g_\phi} = 0, \end{aligned}$$

where  $\nu$  is the unit normal to  $\partial\Omega$  and  $d\operatorname{vol}_{g_\phi}$  is defined according to (3.7).

*Proof.* Just observe that the fact that  $(\nabla u \cdot \nabla \phi) \nabla u \cdot \nu = |\nabla u|^2 \nabla \phi \cdot \nu$  since  $u$  vanishes on  $\partial\Omega$ .  $\square$

We apply the previous analysis to the function

$$\phi := \frac{1}{2} \text{dist}(\cdot, \Lambda)^2,$$

and to the domain  $\Omega = B_\epsilon(\Lambda)$ .

We will need the :

**Lemma 8.3.** *There exists a constant  $C > 0$  such that the following estimates hold in  $B_\epsilon(\Lambda)$*

$$|\Delta\phi - n| \leq C\epsilon \quad \text{and} \quad |\nabla\Delta\phi| \leq C.$$

Moreover, for any  $C^1$  function  $v$  defined on  $B_\epsilon(\Lambda)$

$$\left| \frac{1}{n} \Delta\phi |\nabla v|^2 - \nabla^2\phi(\nabla v, \nabla v) \right| \leq C\epsilon |\nabla v|^2.$$

*Proof.* Follows at once from the expansion of the metric in Fermi coordinates, namely exploiting Lemma 3.1 and Lemma 3.2, and the fact that in Fermi coordinates we have

$$\phi := \frac{1}{2} \text{dist}(\cdot, \Lambda)^2 = \frac{|z|^2}{2}.$$

The lemma follows.  $\square$

Now, Poincaré inequality in  $B_\epsilon(\Lambda)$  reads

**Lemma 8.4.** *There exists a constant  $C > 0$ , such that, for all  $\epsilon \in (0, 1)$  and all  $u \in H_0^1(B_\epsilon(\Lambda))$ , we have*

$$(8.33) \quad \int_{B_\epsilon(\Lambda)} u^2 \text{dvol}_{g_0} \leq C\epsilon^2 \int_{B_\epsilon(\Lambda)} |\nabla u|^2 \text{dvol}_{g_0}.$$

*Proof.* This follows at once from the fact that the Poincaré inequality in the unit ball of  $\mathbb{R}^n$  reads

$$\int_{B_1} u^2 \text{d}z \leq C \int_{B_1} |\nabla u|^2 \text{d}z,$$

and a scaling argument implies that

$$\int_{B_\epsilon} u^2 \text{d}z \leq C\epsilon^2 \int_{B_\epsilon} |\nabla u|^2 \text{d}z.$$

Now, using the product metric  $\bar{g}$  on  $B_\epsilon(\Lambda)$ , this implies that

$$\int_{B_\epsilon(\Lambda)} u^2 \text{dvol}_{\bar{g}} \leq C\epsilon^2 |\Lambda| \int_{B_\epsilon(\Lambda)} |\nabla u|_{g_z}^2 \text{dvol}_{\bar{g}}.$$

Finally, since the Euclidean metric and the product metrics are equivalent (see (3.9)), we conclude that

$$\int_{B_\epsilon(\Lambda)} u^2 \text{dvol}_{g_0} \leq C'\epsilon^2 |\Lambda| \int_{B_\epsilon(\Lambda)} |\nabla u|_{g_0}^2 \text{dvol}_{g_0}.$$

This completes the proof of (8.33).  $\square$

*Proof of Theorem 1.2.* Using Lemma 8.3 and Lemma 8.4 together with Cauchy-Schwarz inequality we get

$$\left| \frac{1}{p+1} \int_{B_\epsilon(\Lambda)} u \nabla u \cdot \nabla \Delta\phi \text{dvol}_{g_0} \right| \leq C\epsilon \int_{B_\epsilon(\Lambda)} |\nabla u|^2 \text{dvol}_{g_0}.$$

Collecting these, together with the result of Lemma 8.2, we conclude that there exists a constant  $C > 0$  such that

$$\left( \frac{n-2}{2} - \frac{n}{p+1} + C\epsilon \right) \int_{B_\epsilon(\Lambda)} |\nabla u|^2 \, \text{dvol}_{g_0} \leq 0,$$

since  $\nabla \phi \cdot \nu = \epsilon$  on  $\partial B_\epsilon(\Lambda)$ . This implies that  $u \equiv 0$  provided  $\epsilon$  is close enough to 0 and  $p > \frac{n+2}{n-2}$ .  $\square$

**Remark 8.1.** *To avoid technical complications we have chosen to carry out the proofs of this section only in the case of a power type nonlinearity. However the same technique can be easily extended to more general nonlinearities.*

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